Hadronic contributions to the anomalous magnetic moment of the muon by QCD model with infinite number of vector mesons

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Abstract

We computed the hadronic vacuum-polarization contributions to the muon anomalous magnetic moment $a_\mu(hadr.)$ by using the QCD model with infinite number of vector mesons [1,2]

The result is $a_\mu(hadr.) = 663(23) \times 10^{-10}$.

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The largest uncertainties in the existing theoretical calculations of the gyromagnetic ratio of the muon $a_\mu = (g-2)/2$ come from the order - α^2 the hadronic contributions to the photon vacuum polarization. In the present paper we suggest the simple method of the calculation of the hadronic vacuum polarization contributions $a_\mu(hadr.)$ by using the QCD model with an infinite number of vector mesons [1,2].

The value of $a_\mu(hadr.)$ can be written as [3]

$$a_\mu(hadr.) = \frac{\alpha^2}{3\pi^2} \int_{4m_\pi^2}^{\infty} ds K(s) R(s)/s \tag{1}$$

where:

$$K(s) = x^2(1-x^2/2) + (1+x)^2(1+x^{-2})\left[\ln(1+x) - x + x^2/2\right] + \\ + \frac{1+x}{1-x} x^2 \ln x \quad ; \quad x = \frac{1-(1-4m_\mu^2/s)^{1/2}}{1+(1-4m_\mu^2/s)^{1/2}} \tag{2}$$

where: m_μ is the muon mass.

The function $R(s)$ is

$$R(s) = \frac{\sigma^H(s)}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{3}$$

where σ^H represents $\sigma(e^+e^- \to hadrons)$ and

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s} \tag{4}$$

Let us write the function $R(s)$ in the form:

$$R = R_\rho + R_\omega + R_s + R_c + R_b$$

Where R_ρ and R_ω are the contributions of u and d quarks in the state with isotopic spin $I = 1$ (ρ - family) and $I = 0$ (ω - family) and R_s, R_c, R_b are the contributions of s (φ - family), c (J/ψ - family) and b (Υ - family) quarks respectively.

Let us consider for example J/ψ family. In the approximation of an infinite number of narrow resonances, having masses M_k and electronic widths Γ_k^{ee} , the function $R_c(s)$ has form:

$$R_c(s) = \frac{9\pi}{\alpha^2} \sum_{k=0}^{\infty} \Gamma_k^{ee} M_k \ \delta(s - M_k^2) \tag{5}$$

Where $\alpha^{-1} = 137.0359895$ [4].

The contribution of R_c in $a_\mu(hadr.)$ has the form:

$$a_\mu^c = \frac{3}{\pi} \sum_{k=0}^{\infty} f(s_k) \ \ , \ \ f(s_k) = \frac{\Gamma_k^{ee} K(s_k)}{M_k} \ \ , \ \ s_k = M_k^2 \tag{6}$$

If for $k > 5$ the total widths Γ_k and masses M_k of the vector mesons obey the conditions

$$M_k^2 - M_{k-1}^2 \ll M_k \Gamma_k \ll M_k^2 \tag{7}$$

then for $k > 5$ the function $R_c(s)$ will be described by a smooth curve and describes an experimental data. All the formulae of the model [5] can be used.

We transform the sum in Eq.(5) into an integral by means of the Euler-Maclaurin formula [6] beginning from $k = k_0 = 4$.

$$\sum_{k=4}^{\infty} f(s_k) = I + \frac{1}{2} f(s_4) - \frac{1}{12} f^{(1)}(s_4) + \frac{1}{720} f^{(3)}(s_4) - \\ - \frac{1}{30240} f^{(5)}(s_4) + \cdots \tag{8}$$

In (8) we have introduced the notations

$$f^{(l)}(s_4) = \frac{\partial^l f(s_k)}{\partial k^l} \ |_{k=4}, \ I = \int_{s_4}^{\infty} f(s_k) \frac{dk}{ds_k} ds_k \tag{9}$$

In Ref. [1,2] we established a correspondence between the electronic width of k-th resonance Γ_k^{ee} and a derivative of the mass of k-th resonance M_k with respect to the number of this resonance

$$\Gamma_k^{ee} = \frac{2\alpha^2}{9\pi} R_c^{PT}(s_k) \frac{dM_k}{dk} \tag{10}$$

The function R_c^{PT} includes all corrections on α_s in perturbation theory (PT).

The term $1/12 f^{(1)}(s_4)$ in Eq.(8) is approximately equal to

$$\frac{1}{12} f^{(1)}(s_4) = \frac{1}{24}\left[f(s_5) - f(s_3)\right] \tag{11}$$

This term is small (see table 2, additional term). The remaining terms in (8) $1/720\ f^{(3)}(s_4) - 1/30240\ f^{(5)}(s_4) + \cdots$ may be omitted due to their smallness. The quantity of a_μ^c practically does not change if $k_0 = 1, 2, 3$. We can not estimate of the term $1/12\ f^{(1)}(s_{k_0})$ if $k_0 = 0$ or 5 .

Integral I in Eq.(9), after the substitution (10), is equal to

$$I = \frac{\alpha^2}{9\pi} \int_{s_4}^{\infty} \frac{K(s) R_c^{PT}(s) ds}{s} \tag{12}$$

where R_c^{PT} we use the formula [1,7]:

$$R_c^{PT}(s) = R_c^{(0)}(s) \mathcal{D}(s) \tag{13}$$

where

$$R_c^{(0)}(s) = \frac{3}{2} Q_c^2 v(3 - v^2)\ ,\ v = \sqrt{1 - \frac{4m_c^2}{s}}\ ,\ Q_c = \frac{2}{3} \tag{14}$$

$m_c = 1.30 \pm 0.05\ GeV$, [2] and

$$\mathcal{D}(v) = \frac{4\pi\alpha_s/3v}{1 - \exp(-4\pi\alpha_s/3v)} - \frac{1}{3}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right)(3 + v)\alpha_s(s) \tag{15}$$

In Eq.(13) we took into account the terms of the first order in α_s and "Coulomb" terms of all orders in α_s/v . At $v \to 1$

$$\mathcal{D}(s) \to 1 + \frac{\alpha_s(s)}{\pi} \tag{16}$$

Note that the function $\mathcal{D}(s)$ differs from (16) strongly in the region of resonances (for example $\mathcal{D}(s_4) = 1.34$ for J/ψ family, $\mathcal{D}(s_4) = 1.59$ for Υ family), hence it is necessary to take into account the "Coulomb" term. For $\alpha_s(s)$ we used formulae [8] described the evolution of $\alpha_s(s)$ taking into account effects of flavor thresholds and the new result - $\alpha_s(m_Z^2) = 0.125 \pm 0.005$ [9]. Within the framework of modified minimal substruction MS scheme [10,11] we obtained $\Lambda^{(5)}_{\overline{MS}} = 0.31(8)$, $\Lambda^{(4)}_{\overline{MS}} = 0.42(9)$, $\Lambda^{(3)}_{\overline{MS}} = 0.46(9)$.

Final formulae for a_μ^c has the form:

$$a_\mu^c = a_\mu^c(Reson.) + a_\mu^c(Int.) + a_\mu^c(Add.) \tag{17}$$

Where:

$$a_\mu^c(Reson.) = \frac{3}{\pi}\left[\sum_{k=0}^{3} \frac{\Gamma_k^{ee} K(s_k)}{M_k} + \frac{\Gamma_4^{ee} K(s_4)}{2M_4}\right]$$

$$a_\mu^c(Int.) = \frac{\alpha^2}{3\pi^2} \int_{s_4}^{\infty} \frac{K(s) R_c^{PT}(s) ds}{s}$$

$$a_\mu^c(Add.) = \frac{1}{8\pi}\left(\frac{\Gamma_3^{ee}}{M_3} - \frac{\Gamma_5^{ee}}{M_5}\right)$$

The formulae for the contribution of the Υ - family in a_μ^b are derived if we replace the index c by b ($m_b = 4.54 \pm 0.02\ GeV$) [2]. The results of the calculations are given in Tables 2,3.

The main contribution to $a_\mu(hadr.)$ comes from the vector mesons consisting of light quarks. First we consider ρ family ($Q_\rho^2 = 1/2$).

Instead of Eq.(17) we have formulae:

$$a_\mu^\rho = a_\mu^\rho(Reson.) + a_\mu^\rho(Int.) + a_\mu^\rho(Add.) \tag{18}$$

Where:

$$a_\mu^\rho(Reson.) = \frac{3}{\pi}\left[\frac{\Gamma_0^{ee}K(s_0)}{M_0} + \frac{\Gamma_1^{ee}K(s_1)}{2M_1}\right]$$

$$a_\mu^\rho(Int.) = \frac{\alpha^2 Q_\rho^2}{3\pi^2}\int_{s_1}^{\infty}\frac{K(s)R^{PT}(s)ds}{s}; \quad Q_\rho^2 = 1/2$$

$$a_\mu^\rho(Add.) = \frac{1}{8\pi}\left(\frac{\Gamma_0^{ee}}{M_0} - \frac{\Gamma_2^{ee}}{M_2}\right)$$

In Eq.(18) the replacement of the summation by an integraction has been carried out starting from $k = 1$, and all terms not written have been ignored.

For $R^{PT}(s)$ we use here the formula [12,13] :

$$R_\rho^{PT}(s) = 3Q_\rho^2\left[1 + \frac{\alpha_s(s)}{\pi} + r_1\left(\frac{\alpha_s(s)}{\pi}\right)^2 + r_2\left(\frac{\alpha_s(s)}{\pi}\right)^3\right] + o(\alpha_s^4(s)) \tag{19}$$

Where:

$$r_1 = 1.9857 - 0.1153f$$

$$r_2 = -6.6368 - 1.2001f - 0.0052f^2 - 1.2395\frac{(\sum_f Q_f)^2}{3\sum_f Q_f^2}$$

Similar equations are valid for a_μ^ω ($Q_\omega^2 = 1/18, \omega$ - family) and a_μ^s ($Q_s^2 = 1/9, \varphi$ - family). Masses and electronic widths of the ρ, ω and φ resonances are presented in Table 1. Note that in the ω case we have two different sets for the describtion of the experimental results [14]. The difference for both cases in value a_μ^ω is negligible.

The results of the calculations are given in Tables 2,3. Summing up the contributions of all families we get:

$$a_\mu(hadr) = 663(23) \times 10^{-10} \tag{20}$$

Note that the QCD model with an infinite number of vector mesons makes possible the calculation of the hadronic contribution to the anomalous magnetic momentum of the muon with the accuracy of 3%. Also this model made possible the calculation of the hadronic contribution to the electromagnetic coupling constant $\alpha(q^2)$ at $q^2 = m_Z^2$ with an accuracy better than 1% [15].

Ths result (20) is in good agreement with $a_\mu(hadr) = 707(6)(17) \times 10^{-10}$ [16] , $a_\mu(hadr) = 710(10.5)(4.9) \times 10^{-10}$ [17] and with earlier works [18,19], obtained by integration formula (1) with the experimental cross-section for e^+e^- annihilation into hadrons.

Summing up the contribution (20) the small rest hadronic contribution $\Delta_\mu(hadr.) = -4.07(0.71)\times 10^{-10}$ [16] the QED contribution $a_\mu(QED) = 1165846984(17)(28)\times 10^{-12}$ [20] and the weak-interaction contribution $a_\mu(weak) = 19.5(0.1) \times 10^{-10}$ [21] we get:

$$a_\mu(Theor.) = 11659148(23) \times 10^{-10} \tag{21}$$

in good agreement with the experimental value [22]

$$a_\mu(Exper.) = 11659230(80) \times 10^{-10} \tag{22}$$

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Table 1. The values of masses and electronic widths of resonances.

u, d - ρ family [4,14]

	0	1	2
$M_{i,Exp.}$ [GeV]	0.7681(5)	1.463(25)	1.73(3)
$\Gamma^{ee}_{i,Exp.}$ [KeV]	6.77(32)	2.5(9)	0.69(15)

u, d - ω family [4,14], two variants

	0	1	2
$M_{i,Exp.}$ [GeV]	0.78195(14)	1.44(7)	1.606(9)
$\Gamma^{ee}_{i,Exp.}$ [KeV]	0.60(2)	0.150(38)	0.140(35)
$M_{i,Exp.}$ [GeV]	0.78195(14)	1.628(14)	-
$\Gamma^{ee}_{i,Exp.}$ [KeV]	0.60(2)	0.37(10)	-

s - φ family [4]

	0	1
$M_{i,Exp.}$ [GeV]	1.019413(8)	1.70(2)
$\Gamma^{ee}_{i,Exp.}$ [KeV]	1.37(5)	0.70(18)

c - J/ψ family [4]

	0	1	2	3	4	5
$M_{i,Exp.}$ [GeV]	3.09693(9)	3.6860(1)	3.7699(25)	4.04(1)	4.159(20)	4.415(6)
$\Gamma^{ee}_{i,Exp.}$ [KeV]	5.36(29)	2.14(21)	0.26(4)	0.75(15)	0.77(23)	0.47(10)

b - Υ family [4]

	0	1	2	3	4	5
$M_{i,Exp.}$ [GeV]	9.46032(22)	10.02330(31)	10.3553(5)	10.5800(35)	10.865(8)	11.019(8)
$\Gamma^{ee}_{i,Exp.}[KeV]$	1.34(4)	0.56(9)	0.44(4)	0.24(5)	0.31(7)	0.13(3)

Table 2. The contributions to $a_\mu(hadr)$

	u, d - ρ	u, d - ω	s - φ	c - J/ψ	b - Υ
Resonances	479(22)	40(1)	44(2)	8(0)	0(0)
Integral	51(01)	6(0)	8(0)	6(0)	0(0)
Add. term	19(00)	2(0)	0(0)	0(0)	0(0)
	549(23)	48(1)	52(2)	14(0)	0(0)

Table 3. Summarized result

	$a_\mu(hadr)$
u, d - ρ	549(23)
u, d - ω	48(01)
s - φ	52(02)
c - J/ψ	14(00)
b - Υ	0(00)
$a_\mu(hadr)$	663(23)